

Constancy of curvature and conformal-projective flatness of statistical manifolds

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Abstract

An identity of conformal-projective curvature tensor of a statistical manifold (M, g, ∇) is studied in this paper. The relation between the constancy of curvature and conformal-projective flatness of statistical manifolds is also discussed.

Keywords: statistical manifold of constant curvature, conformal-projective flatness of statistical manifolds, conformal-projective curvature tensor

1 Introduction

Conformal-projective equivalence of statistical manifolds can be considered as a natural generalization of conformal equivalence of Riemannian metrics, which was introduced by Matsuzoe [4].

Conformal equivalence of Riemannian metrics and projective equivalence of affine connections are combined or generalized to lead up to conformal-projective equivalence or α -conformal equivalence for statistical manifolds (See [2, 4, 5]). Conformal-projective curvature tensor in a statistical manifold plays an important role as Weyl conformal curvature tensor does in Riemannian geometry.

Let M be an n -dimensional manifold, ∇ a torsion-free affine connection on M , and g a Riemannian metric on M .

We denote by $\Gamma(E)$ the set of smooth sections of a vector bundle $E \rightarrow M$. So $\Gamma(TM)$ means the set of smooth vector field on M and $\Gamma(TM^{(r,s)})$ means the set of tensor fields of type (r, s) on M . The curvature tensor of ∇ is denoted by $R \in \Gamma(TM^{(1,3)})$.

Two statistical manifold (M, g, ∇) and $(M, \bar{g}, \bar{\nabla})$ are said to be conformally-projectively equivalent (or generalized conformal equivalent) if there exist two functions $\varphi, \psi \in C^\infty(M)$ satisfying that

$$\begin{aligned}\bar{g}(X, Y) &= e^{\varphi+\psi} g(X, Y) \\ \bar{\nabla}_X Y &= \nabla_X Y + d\varphi(X)Y + d\varphi(Y)X - g(X, Y)\text{grad}_g\psi\end{aligned}$$

for all $X, Y \in \Gamma(TM)$ (See [4]).

Conformal-projective curvature tensor W is defined by

$$\begin{aligned}W(X, Y)Z &= R(X, Y)Z + \frac{1}{n(n-2)}\{Y[(n-1)Ric(X, Z) + {}^*Ric(X, Z)] - \\ &\quad - X[(n-1)Ric(Y, Z) + {}^*Ric(Y, Z)] + [(n-1){}^*Ric(Y) + Ric^\sharp(Y)]g(X, Z) - \\ &\quad - [(n-1){}^*Ric(X) + Ric^\sharp(X)]g(Y, Z)\} + \frac{\sigma}{(n-1)(n-2)}[Xg(Y, Z) - Yg(X, Z)]\end{aligned}\quad (1.1)$$

and an $n(\geq 4)$ -dimensional statistical manifold (M, g, ∇) is conformally-projectively flat if and only if the conformal-projective curvature tensor vanishes everywhere on M , where R, Ric and σ are curvature tensor field, Ricci tensor field and scalar curvature of (M, g, ∇) , respectively, and the Ricci operator Ric^\sharp of (M, g, ∇) is the $(1, 1)$ -tensor field determined by

$$g(Ric^\sharp(X), Y) = Ric(X, Y)$$

and the corresponding quantities for $\bar{\nabla}^*$ which is a dual connection of ∇ are denoted with $*$ (See [3]).

On the other hand, for any $\alpha \in \mathbf{R}$, two statistical manifolds (M, g, ∇) and $(M, \bar{g}, \bar{\nabla})$ are said to be α -conformally equivalent if there exists a function $\varphi \in C^\infty(M)$ satisfying that

$$\begin{aligned} \bar{g}(X, Y) &= e^\varphi g(X, Y) \\ g(\bar{\nabla}_X Y, Z) &= g(\nabla_X Y, Z) - \frac{1+\alpha}{2} d\varphi(Z)g(X, Y) + \frac{1-\alpha}{2} [d\varphi(X)g(Y, Z) + d\varphi(Y)g(X, Z)] \end{aligned}$$

for all $X, Y, Z \in \Gamma(TM)$ (See [2]).

It is easily verified that if two statistical manifolds (M, g, ∇) and $(M, \bar{g}, \bar{\nabla})$ are 1-conformal equivalent then they are conformal-projective equivalent and so if a statistical manifold (M, g, ∇) is 1-conformal flat then it is conformal-projective flat.

We study some properties of conformal-projective curvature tensor of a statistical manifold and a sufficient condition for a statistical manifold to be conformal-projective flat in this paper.

In section 2, we show some properties of conformal-projective curvature tensor of a statistical manifold.

In section 3, we show that a statistical manifold of constant curvature is conformal-projective flat.

2 Conformal-projective curvature tensor of a statistical manifold

In this section, we give an expression of conformal-projective curvature tensor W of a statistical manifold (M, g, ∇) and show the relationship between conformal-projective curvature tensor W of a statistical manifold (M, g, ∇) and one \bar{W}^* of a dual statistical manifold $(M, g, \bar{\nabla}^*)$.

We first give the following fact, which has been obtained independently by Zhang [6]. We quote the fact from his paper with a suitable modification for later use.

Proposition 2.1 ([6]) *Let σ and $\bar{\sigma}^*$ be the scalar curvature of a statistical manifold (M, g, ∇) and a dual statistical manifold $(M, g, \bar{\nabla}^*)$, respectively. Then we have*

$$\sigma = \bar{\sigma}^* \quad (2.1)$$

Lemma 2.1 *The conformal-projective curvature tensor W of a statistical manifold (M, g, ∇) can be expressed as follows:*

$$W(X, Y)Z = R(X, Y)Z + YL(X, Z) - XL(Y, Z) + \bar{L}^{\sharp*}(Y)g(X, Z) - \bar{L}^{\sharp*}(X)g(Y, Z) \quad (2.2)$$

for all $X, Y, Z \in \Gamma(TM)$, where L, \bar{L}^* and $\bar{L}^{\sharp*}$ are tensor fields of type $(0, 2)$ and $(1, 1)$, respectively, given by

$$\begin{aligned}
L(X, Y) &= \frac{1}{n-2} \left\{ \frac{1}{n} [(n-1) Ric(X, Y) + {}^* Ric(X, Y)] - \frac{\sigma}{2(n-1)} g(X, Y) \right\} \\
{}^* L(X, Y) &= \frac{1}{n-2} \left\{ \frac{1}{n} [(n-1) {}^* Ric(X, Y) + Ric(X, Y)] - \frac{{}^* \sigma}{2(n-1)} g(X, Y) \right\} \\
g({}^* L(X), Y) &= {}^* L(X, Y)
\end{aligned}$$

Proof. Using Eq. (2.1), we can express Eq. (1.1) as follows:

$$\begin{aligned}
W(X, Y)Z &= R(X, Y)Z + \frac{1}{n(n-2)} \{ Y[(n-1) Ric(X, Z) + {}^* Ric(X, Z)] - \\
&\quad - X[(n-1) Ric(Y, Z) + {}^* Ric(Y, Z)] + [(n-1) {}^* Ric^\sharp(Y) + Ric^\sharp(Y)]g(X, Z) - \\
&\quad - [(n-1) {}^* Ric^\sharp(X) + Ric^\sharp(X)]g(Y, Z) \} + \frac{\sigma + {}^* \sigma}{2(n-1)(n-2)} [Xg(Y, Z) - Yg(X, Z)] \\
&= R(X, Y)Z + \frac{1}{n-2} Y \left\{ \frac{1}{n} Ric(Y, Z) + {}^* Ric(X, Z) \right\} - \frac{\sigma}{2(n-1)} g(X, Z) - \\
&\quad - \frac{1}{n-2} X \left\{ \frac{1}{n} [(n-1) Ric(Y, Z) + {}^* Ric(Y, Z)] - \frac{\sigma}{2(n-1)} g(Y, Z) + \right. \\
&\quad + \frac{1}{n-2} \left\{ \frac{1}{n} [(n-1) {}^* Ric^\sharp(Y) + Ric^\sharp(Y)] - \frac{{}^* \sigma}{2(n-1)} g(X, Z) + \right. \\
&\quad + \left. \frac{1}{n-2} \left\{ \frac{1}{n} [(n-1) {}^* Ric^\sharp(X) + Ric^\sharp(X)] - \frac{{}^* \sigma}{2(n-1)} g(Y, Z) \right\} \right\}
\end{aligned}$$

On the other hand, since $g({}^* Ric^\sharp(X), Y) = {}^* Ric(X, Y)$ and $g(Ric^\sharp(X), Y) = Ric(X, Y)$ hold for all $X, Y \in \Gamma(TM)$, we have

$$g((n-1) {}^* Ric^\sharp(X) + Ric^\sharp(X) - \frac{{}^* \sigma}{2(n-1)} X, Y) = {}^* L(X, Y)$$

for all $X, Y \in \Gamma(TM)$. So

$${}^* L(X) = (n-1) {}^* Ric^\sharp(X) + Ric^\sharp(X) - \frac{{}^* \sigma}{2(n-1)} X$$

holds for all $X \in \Gamma(TM)$. Therefore we have

$$W(X, Y)Z = R(X, Y)Z + YL(X, Z) - XL(Y, Z) + {}^* L^\sharp(Y)g(X, Z) - {}^* L^\sharp(X)g(Y, Z)$$

for all $X, Y, Z \in \Gamma(TM)$. \square

Theorem 2.1 Let W and ${}^* W$ be the conformal-projective curvature tensors of a statistical manifold (M, g, ∇) and a dual statistical manifold $(M, g, {}^* \nabla)$, respectively. Then we have

$$g(W(X, Y)Z, U) + g({}^* W(X, Y)U, Z) = 0 \quad (2.3)$$

for all $X, Y, Z, U \in \Gamma(TM)$.

Proof. From Eq. (2.2), we have

$$\begin{aligned} g(W(X, Y)Z, U) &= g(R(X, Y)Z, U) + g(Y, U)L(X, Z) - g(X, U)L(Y, Z) + \\ &\quad + \overset{*}{L}(Y, U)g(X, Z) - \overset{*}{L}(X, U)g(Y, Z) \\ g(\overset{*}{W}(X, Y)U, Z) &= g(\overset{*}{R}(X, Y)U, Z) + g(Y, Z)\overset{*}{L}(X, U) - g(X, Z)\overset{*}{L}(Y, U) + \\ &\quad + L(Y, Z)g(X, U) - L(X, Z)g(Y, U) \end{aligned}$$

for all $X, Y, Z, U \in \Gamma(TM)$. Since $g(R(X, Y)Z, U) + g(\overset{*}{R}(X, Y)U, Z) = 0$ holds for all $X, Y, Z, U \in \Gamma(TM)$, Eq. (2.3) holds. \square

Eq. (2.3) shows that a statistical manifold (M, g, ∇) is conformally-projectively flat if and only if the dual statistical manifold $(M, g, \overset{*}{\nabla})$ is conformally-projectively flat.

3 Constancy of curvature and conformal-projective flatness of a statistical manifold

It is known that if an $n(\geq 4)$ -dimensional Riemannian manifold is of constant curvature, it is conformal flat and that an $n(\geq 3)$ -dimensional Riemannian manifold is projective flat if and only if it is of constant curvature.

The conformal-projective equivalence of a statistical manifold is a generalization of conformal equivalence and projective equivalence of a Riemannian manifold. So constancy of curvature of a statistical manifold has a close relationship to the conformal-projective equivalence of a statistical manifold. It is shown that an $n(\geq 2)$ -dimensional statistical manifold (M, g, ∇) is of constant curvature if and only if the tangent bundle TM over M with complete lift statistical structure (g^c, ∇^c) is conformally-projectively flat by Hasegawa [1].

The following theorem shows that the relationship between constancy of curvature and conformal-projective flatness of a statistical manifold (M, g, ∇) .

Theorem 3.1 *If an $n(\geq 4)$ -dimensional statistical manifold (M, g, ∇) is of constant curvature, it is conformally-projectively flat.*

Proof. From Eq. (2.2), we have

$$W(X, Y)Z = R(X, Y)Z + YL(X, Z) - XL(Y, Z) + \overset{*}{L}^\#(Y)g(X, Z) - \overset{*}{L}^\#(X)g(Y, Z)$$

for all $X, Y, Z \in \Gamma(TM)$.

Since a statistical manifold (M, g, ∇) is of constant curvature,

$$Ric = \overset{*}{Ric}$$

holds and so we have

$$L(X, Y) = \overset{*}{L}(X, Y) = \frac{1}{n-2} \{ Ric(X, Y) - \frac{\sigma}{2(n-1)} g(X, Y) \}$$

for all $X, Y \in \Gamma(TM)$. On the other hands, since

$$R(X, Y)Z = K\{g(Y, Z) - g(X, Z)\}$$

holds for all $X, Y, Z \in \Gamma(TM)$, we have

$$Ric(Y, Z) = tr\{X \mapsto R(X, Y)Z\} = (n-1)Kg(Y, Z)$$

for all $Y, Z \in \Gamma(TM)$. Since from the above equation,

$$\sigma = tr_g\{(Y, Z) \mapsto Ric(Y, Z)\} = n(n-1)K$$

holds, we have

$$L(X, Y) = L^*(X, Y) = \frac{K}{2}g(X, Y)$$

for all $X, Y \in \Gamma(TM)$ and so

$$L^\sharp(X) = L^{*\sharp}(X) = \frac{K}{2}X$$

holds for all $X \in \Gamma(TM)$. Therefore we have

$$\begin{aligned} W(X, Y)Z &= R(X, Y)Z + YL(X, Z) - XL(Y, Z) + L^*(Y)g(X, Z) - L^*(X)g(Y, Z) \\ &= K\{Xg(Y, Z) - Yg(X, Z)\} + \frac{K}{2}Yg(X, Z) - \frac{K}{2}Xg(Y, Z) + \\ &\quad + \frac{K}{2}Yg(X, Z) - \frac{K}{2}Xg(Y, Z) \\ &= 0 \end{aligned}$$

for all $X, Y, Z \in \Gamma(TM)$. So the proof is finished. \square

If a statistical manifold is a self-dual statistical manifold, that is, a Riemannian manifold, conformal-projective flatness of a statistical manifold becomes to conformal flatness of a Riemannian manifold. So theorem 3.1 shows that if a Riemannian manifold is of constant curvature, it is conformal flat, which is well known in Riemannian geometry.

Consequently, theorem 3.1 generalizes the fact that an $n(\geq 4)$ -dimensional Riemannian manifold of constant curvature is conformal flat to case of a statistical manifold.

Theorem 3.1 and theorem in [1] give the following:

Corollary 3.1 *Let (M, g, ∇) be an $n(\geq 4)$ -dimensional statistical manifold. If the tangent bundle (TM, g^c, ∇^c) over M with complete lift statistical structure is conformally-projectively flat, (M, g, ∇) is conformally-projectively flat.*

References

- [1] I. Hasegawa, K. Yamauchi, Conformal-projective flatness of tangent bundle with complete lift statistical structure, Differential Geometry-Dynamical Systems, 10, 148-158, 2008.
- [2] T. Kurose, On the divergence of 1-conformally flat statistical manifolds, Tôhoku Math. J., 46, 427-433, 1994.
- [3] T. Kurose, Conformal-Projective geometry of Statistical Manifolds, Interdisciplinary information Sciences, 8, 1, 89-100, 2002.
- [4] H. Matsuzoe, On realization of conformally-projectively flat statistical manifolds and the divergences, Hokkaido Math. J., 27, 409-421, 1998.

- [5] K. Uohashi, On α -conformal equivalence of statistical manifolds, J. Geom., 75, 179-184, 2002.
- [6] J. Zhang, A note on curvature of α -connections of a statistical manifold, AISM, 59, 161-170, 2007.